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BIFURCATION CONDITIONS FOR IDEAL FIBRE-REINFORCED MATERIALS

M. ARCISZ
Polish Academy of Sciences, Institute of Fundamental Technological Research, Swietokrzyska 21, 00.049 Warsaw, Poland

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Abstract—A bifurcation of an equilibrium state for ideal fibre-reinforced material is discussed. It is assumed that the material is elastic, locally transversely isotropic, incompressible and inextensible in the direction of fibres. On a finite state of strain an arbitrary field of small displacements is superposed and a set of governing equations for the perturbed state is derived.

As an example a stability problem of a rectangular block subjected to a finite, homogeneous deformation is considered. A discussion of the results is focused on the influence on the stability of the pressure applied in the direction of fibres.

Due to the assumption of inextensibility this pressure has no influence on the state of strain, but it is shown that it may cause a loss of stability.

INTRODUCTION

The problem of bifurcation of an equilibrium state for elastic, homogeneous, both isotropic and anisotropic bodies has been discussed in numerous papers, e.g. [2–7]. The bifurcation conditions are derived on the basis of the theory of superposition of small elastic deformations on finite deformations, formulated by Green, Rivlin and Shield[2]. The problem concerns establishing the range of loads for which non-zero superposed deformations are admissible. It is assumed that these loads may induce a loss of stability and due to this assumption a state of body for which there exists a non-zero solution for superposed deformations is called unstable.

The purpose of this paper is to investigate the bifurcation conditions for ideal fibre-reinforced materials. In the theory of ideal fibre-reinforced materials developed by Spencer[1] it is assumed that a vector **A** tangent to fibres is defined at each point of the medium and the material is inextensible in the **A**-direction. It is further assumed, that the material is incompressible and locally transversely isotropic, with **A** as the local isotropy axis. We shall consider finite eleasticity assuming that the strain energy is the potential for stresses.

In Section 1 we state the governing equations for the considered material using convected coordinates. In Section 2 a perturbed state is defined, and the equations which must be satisfied by the increments of the considered quantities are derived. The results obtained in Sections 1 and 2, are applied to the stability problem of a rectangular block subjected to a finite, homogeneous deformation. Section 3 contains the solution of the problem, and the results are discussed in Sections 4–6

1. BASIC EQUATIONS

Let us introduce two Cartesian coordinate systems Z^α and z^i with base vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_i$, respectively. In the undeformed state B_0, the position of a material point is given by three functions

$$Z^\alpha = Z^\alpha(\Theta^K). \tag{1.1}$$

A body subjected to a finite strain passes from the state B_0 to a state B and a material point Θ^K at time t has the position

$$z^i = z^i(\Theta^K, t). \tag{1.2}$$

Denoting by $\mathring{\mathbf{r}}$ and $\mathbf{r}$ the position vectors of a material point Θ^K in the states B_0 and B, respectively, we introduce the base vectors of the curvilinear, convected coordinates

$$\mathring{\mathbf{g}}_K \equiv \frac{\partial \mathring{\mathbf{r}}}{\partial \Theta^K} = \frac{\partial Z^\alpha}{\partial \Theta^K}\mathbf{e}_\alpha \quad \text{in} \quad B_0, \tag{1.3}$$

$$\mathbf{g}_K \equiv \frac{\partial \mathbf{r}}{\partial \Theta^K} = \frac{\partial z^i}{\partial \Theta^K}\mathbf{e}_i \quad \text{in} \quad B$$

and the metric tensors

$$\mathring{g}_{KL} \equiv \mathring{\mathbf{g}}_K \cdot \mathring{\mathbf{g}}_L = \frac{\partial Z^\alpha}{\partial \theta^K}\frac{\partial Z^\beta}{\partial \Theta^L}\delta_{\alpha\beta} \quad \text{in} \quad B_0, \tag{1.4}$$

$$g_{KL} \equiv \mathbf{g}_K \cdot \mathbf{g}_L = \frac{\partial z^m}{\partial \Theta^K}\frac{\partial z^n}{\partial \Theta^L}\delta_{mn} \quad \text{in} \quad B.$$

Choosing the tensor g_{KL} as the measure of strain the constitutive equations of an elastic incompressible material take the form

$$\tau^{KL} = 2\frac{\partial W}{\partial g_{KL}}, \tag{1.5}$$

where τ^{KL} are the components of the Cauchy stress tensor in the base $\mathbf{g}_K$ and $W = W(g_{KL})$ is the strain energy function measured per unit volume of the undeformed body.

For the ideal fibre-reinforced material (see Ref. [1]) the strain energy is a function of five invariants of the strain tensor and a vector tangent to fibres. Denoting by $\mathring{A}^K$, A^K the components of a vector tangent to a fibre in the bases $\mathring{\mathbf{g}}_K$ and $\mathbf{g}_K$, respectively, we find that

$$\mathring{A}^K = A^K \tag{1.6}$$

since in the convected coordinate system the components of a vector tangent to a material line remain unchanged during deformation. The invariants, in terms of tensor g_{KL} and vector $\mathring{A}^K$, have the form

$$\begin{aligned}
I_1 &= g_{KL}\mathring{g}^{KL},\\
I_2 &= \tfrac{1}{2}g_{KL}g_{MN}(\mathring{g}^{KL}\mathring{g}^{MN} - \mathring{g}^{LM}\mathring{g}^{KN}) = g^{KL}\mathring{g}_{KL}I_3,\\
I_3 &= (\det g_{KL})(\det \mathring{g}_{MN})^{-1}\\
I_4 &= \mathring{A}^K\mathring{A}^N g_{KL}g_{MN}\mathring{g}^{LM},\\
I_5 &= \mathring{A}^K\mathring{A}^L g_{KL}.
\end{aligned} \tag{1.7}$$

The conditions of incompressibility and inextensibility imply that

$$\begin{aligned}
&(\det g_{KL})(\det \mathring{g}_{KL})^{-1} = 1,\\
&\mathring{A}^K\mathring{A}^L g_{KL} = \mathring{A}^K\mathring{A}^L\mathring{g}_{KL} = 1.
\end{aligned} \tag{1.8}$$

Introducing Lagrange multipliers we can express the strain enery function as follows

$$W = \bar{W}(I_1, I_2, I_4) - \tfrac{1}{2}p(I_3 - 1) + \tfrac{1}{2}T(I_5 - 1), \tag{1.9}$$

where p and T are arbitrary scalar functions.

Substituting (1.9) and (1.7) in to (1.5) we obtain

$$\tau^{KL} = \Phi_1\mathring{g}^{KL} + \Phi_2 b^{KL} + \Phi_4 c^{KL} - pg^{KL} + T\mathring{A}^K\mathring{A}^L, \tag{1.10}$$

where

$$\Phi_M \equiv 2\frac{\partial \bar{W}}{\partial I_M},$$
$$b^{KL} \equiv (\mathring{g}^{KL}\mathring{g}^{MN} - \mathring{g}^{LM}\mathring{g}^{KN})g_{MN}, \tag{1.11}$$
$$c^{KL} \equiv (\mathring{A}^K\mathring{A}^N\mathring{g}^{LM} + \mathring{A}^M\mathring{A}^L\mathring{g}^{NK})g_{MN}.$$

The equilibrium equations in the state B are

$$\tau^{KL}|_L = 0, \tag{1.12}$$

where symbol $\ldots|_L$ denotes the covariant derivative with respect to Θ^L in the base $\mathbf{g}_K$.

The stability of a boundary value problem for the equations describing the state B, namely the equations of equilibrium (1.12) and two equations following from the kinematical constraints (1.8), is the problem under consideration.

2. PERTURBED STATE

We assume that the body in the deformed state B undergoes a small displacement $\epsilon\mathbf{w}$ passing to the state B^*. A position vector of a material point Θ^K in state B^* is therefore

$$\overset{*}{\mathbf{r}} = \mathbf{r} + \epsilon\mathbf{w}, \tag{2.1}$$

where ϵ is a small quantity, so that ϵ^2 and its higher powers can be disregarded as compared with ϵ.

Due to the additional displacements, each quantity $\mathbf{d}$ defined for the state B will vary and denoting this increment by prime we have in the state B^*

$$\mathbf{d}^* = \mathbf{d} + \epsilon\mathbf{d}' \tag{2.2}$$

and for example the base vectors in the state B^* are $\overset{*}{\mathbf{g}}_K \equiv \frac{\partial \overset{*}{\mathbf{r}}}{\partial \Theta^K} = \mathbf{g}_K + \epsilon\mathbf{g}'_K$ where

$$\mathbf{g}'_K = w^R|_K\mathbf{g}_R. \tag{2.3}$$

Consequently, we find that

$$\begin{aligned}
g'_{KL} &= w_K|_L + w_L|_K,\\
I'_1 &= g'_{KL}\mathring{g}^{KL},\\
I'_2 &= \mathring{g}_{KL}(g'^{KL}I_3 + g^{KL}I'_3),\\
I'_3 &= I_3 g^{KL}g'_{KL},\\
I'_4 &= \mathring{A}^K\mathring{A}^N\mathring{g}^{LM}(g_{KL}g'_{MN} + g'_{KL}g_{MN}),\\
I'_5 &= \mathring{A}^K\mathring{A}^L g'_{KL},\\
b'^{KL} &= (\mathring{g}^{KL}\mathring{g}^{MN} - \mathring{g}^{LM}\mathring{g}^{KN})g'_{MN},\\
c'^{KL} &= (\mathring{A}^K\mathring{A}^N\mathring{g}^{LM} + \mathring{A}^M\mathring{A}^L\mathring{g}^{NK})g'_{MN}.
\end{aligned} \tag{2.4}$$

The conditions of incompressibility and inextensibility imply that $I_3' = I_5' = 0$. Therefore, the displacements $\mathbf{w}$ are restricted by the constraints

$$\begin{aligned} w^K|_K &= 0, \\ \mathring{A}^K\mathring{A}^L(w_K|_L + w_L|_K) &= 0. \end{aligned} \tag{2.5}$$

Stresses in the state $\overset{*}{B}$ are

$$\overset{*}{\tau}{}^{KL} = \overset{*}{\Phi}_1\overset{*}{g}{}^{KL} + \overset{*}{\Phi}_2\overset{*}{b}{}^{KL} + \overset{*}{\Phi}_4\overset{*}{c}{}^{KL} - \overset{*}{p}\overset{*}{g}{}^{KL} + \overset{*}{T}\mathring{A}^K\mathring{A}^L, \tag{2.6}$$

where

$$\overset{*}{\Phi}_K \equiv 2\frac{\partial \bar{W}(\overset{*}{I}_M)}{\partial \overset{*}{I}_K} = \Phi_K(I_M + \epsilon I_M').$$

Expanding $\overset{*}{\Phi}_K$ into the Taylor series we obtain

$$\Phi_K' = \Phi_{KL}I_L', \qquad \Phi_{KL} \equiv \frac{2\partial^2 \bar{W}}{\partial I_K \partial I_L} \tag{2.7}$$

and finally

$$\tau'^{KL} = \Phi_{1R}I_R'g^{KL} + \Phi_{2R}I_R'b^{KL} + \Phi_{4R}I_R'c^{KL} + \Phi_2 b'^{KL} + \Phi_4 c'^{KL} - pg'^{KL} - p'g^{KL} + T'\mathring{A}^K\mathring{A}^L. \tag{2.8}$$

From the equations of equilibrium in the state $\overset{*}{B}$

$$\overset{*}{\tau}{}^{KR}_{,R} + \overset{*}{\Gamma}{}^K_{RS}\overset{*}{\tau}{}^{RS} + \overset{*}{\Gamma}{}^R_{RS}\overset{*}{\tau}{}^{SK} = 0 \tag{2.9}$$

substituting $\overset{*}{\tau}{}^{RS} = \tau^{RS} + \epsilon\tau'^{RS}$, $\overset{*}{\Gamma}{}^K_{LM} = \Gamma^K_{LM} + \epsilon\Gamma'^K_{LM}$ and using (1.12) we have

$$\tau'^{KR}|_R + w^K|_{RS}\tau^{RS} + w^R|_{RS}\tau^{SK} = 0. \tag{2.10}$$

The eqns (2.10) and (2.5) constitute the set of five linear differential equations for five unknown functions w^K, p', T'.

We say that the state B, defined by a boundary value problem for eqns (1.12), (1.8) is unstable if the corresponding boundary value problem for eqns (2.10), (2.5) has a non-trivial solution $\mathbf{w} \neq 0$.

To formulate static boundary conditions for the perturbed state $\overset{*}{B}$, we express the stress vector $\overset{*}{\mathbf{P}}$, acting on the surface $\overset{*}{S}$, by the already known quantities. We have

$$\overset{*}{P}{}^K = \overset{*}{\tau}{}^{KL}\overset{*}{n}_L, \tag{2.11}$$

where $\overset{*}{n}_L$ are the components of the normal vector to $\overset{*}{S}$ in the base $\overset{*}{g}_K$. Writing $\overset{*}{\mathbf{n}}$ as the vector product of two vectors tangent to materal lines lying on $\overset{*}{S}$ we obtain

$$\overset{*}{n}_L = n_L + \epsilon n_L n^P n^R w_{P|R}. \tag{2.12}$$

For the normal load $\overset{*}{P}_K = -\overset{*}{P}\overset{*}{n}{}^K$ we find from (2.11) for terms independent of ϵ

$$\tau^{PR}n_R g_{PK} = -Pn_K \tag{2.13}$$

and for terms linear with respect to ϵ

$$\tau'^{PR}n_R g_{PK} + \tau^{PR}n_R g'_{PK} = -P'_{n_K}. \tag{2.14}$$

The results obtained in Sections 1 and 2 will be illustrated by an example of rectangular block subjected to finite, homogeneous deformation, emphasizing of the results significant for fibre reinforced materials.

3. STABILITY OF RECTANGULAR BLOCK SUBJECTED TO A HOMOGENEOUS FINITE DEFORMATION

Consider a rectangular block subjected to the homogeneous deformation

$$z^1 = \lambda_1 Z^1, \qquad z^2 = \lambda_2 Z^2, \qquad z^3 = \lambda_3 Z^3 \tag{3.1}$$

and reinforced by one family of straight fibres parallel to Z^1-axis. The vector tangent to a fibre has the components

$$\mathring{A}^K = (1, 0, 0). \tag{3.2}$$

The convected coordinates Θ^K are chosen in such a way that they coincide in the state B with the cartesian coordinates $z^i = (x, y, z)$. In the deformed state the block occupies a region $-l \leq x \leq l$, $-h \leq y \leq h$, $-b \leq z \leq b$.

From (1.4) and (3.1) we obtain

$$g_{KL} = \delta_{KL} \qquad \mathring{g}_{KL} = \begin{bmatrix} (\lambda_1)^{-2} & 0 & 0 \\ 0 & (\lambda_2)^{-2} & 0 \\ 0 & 0 & (\lambda_3)^{-2} \end{bmatrix}. \tag{3.3}$$

The conditions of incompressibility and inextensibility (1.8) imply $\lambda_1\lambda_2\lambda_3 = 1$ and $\mathring{g}_{11} = g_{11}$. Denoting $\lambda \equiv \lambda_2$ we have

$$\lambda_1 = 1, \qquad \lambda_2 = \lambda, \qquad \lambda_3 = \lambda^{-1}. \tag{3.4}$$

The stress tensor (1.10) has the components

$$\begin{aligned} \tau^{11} &= \Phi_1 + \Phi_2(\lambda^2 + \lambda^{-2}) + 2\phi_4 + T - p \\ \tau^{22} &= \Phi_1\lambda^2 + \Phi_2(1 + \lambda^2) - p \\ \tau^{33} &= \Phi_1\lambda^{-2} + \Phi_2(1 + \lambda^{-2}) - p \\ \tau^{12} &= \tau^{13} = \tau^{23} = 0 \end{aligned} \tag{3.5}$$

and from the equilibrium equations (1.12) we obtain

$$p = p(x), \qquad (T - p)_{,x} = 0. \tag{3.6}$$

We shall consider the stability of rectangular block uniformly loaded on its surfaces, i.e. we assume

$$\begin{aligned} \tau^{11}(\pm l, y, z) &\equiv -t = \text{const.} \\ p &= \text{const.} \end{aligned} \tag{3.7}$$

As a consequence of the assumption of inextensibility the pressure t acting in the direction of fibres has no influence on the state of strain. There arise the following questions: (i) has the pressure t an influence on the stability and if so is it stabilizing or destabilizing?, (ii) does there exist a value of t such that for $\lambda = 1$ the specimen is unstable?

In both cases we shall investigate the influence of two parameters: λ and t on the stability of the solution.

Let us superpose on the above defined state B an arbitrary field of small displacements $\epsilon \mathbf{w}$;

$$w^K \equiv (u, v, w). \tag{3.8}$$

In view of (3.3)$_1$ we have $w_K|_L = w_{K,L}$ and from (2.4)$_1$ we obtain

$$g'_{KL} = -g'^{KL} = \begin{bmatrix} 2u_{,x} & v_{,x}+u_{,y} & w_{,x}+u_{,z} \\ u_{,y}+v_{,x} & 2v_{,y} & w_{,y}+v_{,z} \\ u_{,z}+w_{,x} & v_{,z}+w_{,y} & 2w_{,z} \end{bmatrix}. \tag{3.9}$$

The conditions of incompressibility and inextensibility (2.5) imply that

$$\begin{aligned} v_{,y}+w_{,z} &= 0, \\ u_{,x} &= 0. \end{aligned} \tag{3.10}$$

For the increments of stress components determined in (2.8) we obtain

$$\begin{aligned}
\tau'^{11} &= 2v_{,y}(\lambda^2-\lambda^{-2})[\Phi_{11}+\Phi_{12}+(\lambda^2-\lambda^{-2})(\Phi_{12}+\Phi_{22})+2(\Phi_{14}+\Phi_{24})+\Phi_2]+T'-p', \\
\tau'^{22} &= 2v_{,y}\{(\lambda^2-\lambda^{-2})[\Phi_{12}+\Phi_{22}+\lambda^2(\Phi_{11}+2\Phi_{12}+\Phi_{22})]-\Phi_2+p\}-p', \\
\tau'^{33} &= 2v_{,y}\{(\lambda^2-\lambda^{-2})[\Phi_{12}+\Phi_{22}+\lambda^{-2}(\Phi_{11}+2\Phi_{12}+\Phi_{22})]+\Phi_2-p\}-p', \\
\tau'^{12} &= (u_{,y}+v_{,x})[p-\lambda^2(\Phi_2-\Phi_4)], \\
\tau'^{13} &= (u_{,z}+w_{,x})[p-\lambda^{-2}(\Phi_2-\Phi_4)], \\
\tau'^{23} &= (v_{,z}+w_{,y})[p-\Phi_2].
\end{aligned} \tag{3.11}$$

The equations of equilibrium (2.10) can be written as follows

$$\begin{aligned}
a_1(\lambda)u_{,yy}+a_2(\lambda)u_{,zz}+a_3(\lambda)v_{,xy}+(T'-p')_{,x} &= 0, \\
b_1(\lambda,t)v_{,xx}+b_2(\lambda)v_{,yy}+b_3(\lambda)v_{,zz}-p'_{,y} &= 0, \\
c_1(\lambda,t)w_{,xx}+c_2(\lambda)w_{,yy}+c_3(\lambda)w_{,zz}-p'_{,z} &= 0,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
a_1(\lambda) &\equiv \lambda^2(\Phi_1+\Phi_4)+\Phi_2 \\
a_2(\lambda) &\equiv \lambda^{-2}(\Phi_1+\Phi_4)+\Phi_2 \\
a_3(\lambda) &\equiv (\lambda^2-\lambda^{-2})\{\Phi_2+\Phi_4+2[\Phi_{11}+\Phi_{12}+(\lambda^2+\lambda^{-2})(\Phi_{12}+\Phi_{22})+2(\Phi_{14}+\Phi_{24})]\} \\
b_1(\lambda,t) &\equiv p-t-\lambda^2(\Phi_2-\Phi_4) \\
b_2(\lambda) &\equiv \lambda^2(\Phi_1+\Phi_2)+2(\lambda^2-\lambda^{-2})[\Phi_{12}+\Phi_{22}+\lambda^2(\Phi_{11}+2\Phi_{12}+\Phi_{22})] \\
b_3(\lambda) &\equiv \lambda^{-2}(\Phi_1+\Phi_2) \\
c_1(\lambda,t) &\equiv p-t-\lambda^{-2}(\Phi_2-\Phi_4) \\
c_2(\lambda) &\equiv \lambda^2(\Phi_1+\Phi_2) \\
c_3(\lambda) &\equiv \lambda^{-2}(\Phi_1+\Phi_2)-2(\lambda^2-\lambda^{-2})[\Phi_{12}+\Phi_{22}+\lambda^{-2}(\Phi_{11}+2\Phi_{12}+\Phi_{22})].
\end{aligned} \tag{3.13}$$

The equations of equilibrium (3.12) together with eqns (3.10) constitute a set of five differential equations of second order with five unknown functions u, v, w, p', T' of three variables x, y, z and two parameters λ, t.

Prior to writing down the boundary conditions for the above set of equations we have to specify the boundary conditions for the state B. We shall discuss two cases: (i) a rectangular block between two pairs of rigid smooth plates with surfaces $y = \pm h$ free of load, (ii) a rectangular block with rigidly confined boundaries. The latter is briefly discussed in Section 6. Considering the former we shall seek a solution of eqns (3.12), (3.10) with the following boundary conditions: the surfaces $x = \pm l$, $z = \pm b$ are plane and free of tangent load, while surfaces $y = \pm h$ are free of load.

From (2.13) and (2.14) we obtain

$$p = \lambda^2\Phi_1 + (1+\lambda^2)\Phi_2 \tag{3.14}$$

and

$$\begin{aligned}
&\text{on} \quad x = \pm l: \quad u = 0, \quad u_{,y} + v_{,x} = 0, \quad u_{,z} + w_{,x} = 0, \\
&\text{on} \quad y = \pm h: \quad u_{,y} + v_{,x} = 0, \quad v_{,z} + w_{,y} = 0, \quad p' - (b_2 + c_2)v_{,y} = 0, \\
&\text{on} \quad z = \pm b: \quad w = 0, \quad u_{,z} + w_{,x} = 0, \quad v_{,z} + w_{,y} = 0,
\end{aligned} \tag{3.15}$$

where b_2, c_2 are defined in (3.13).

Equations $(3.10)_2$ and $(3.12)_1$ can be directly solved. The first, with the boundary condition $u(\pm l) = 0$, implies that

$$u = 0 \tag{3.16}$$

in the whole region and from the second we find the function $T'(x, y, z)$:

$$T' = p' - a_3(\lambda)v_{,y} + c(y, z), \tag{3.17}$$

where a_3 is given in (3.13) and c is an arbitrary function.

The functions v, w, p' can be expanded into serie, in the orthogonal basis of trigonometric functions; satisfying the boundary conditions (3.15) on $x = \pm l$ and $z = \pm b$ we have

$$\begin{aligned}
v(x, y, z) &= \sum_m \sum_n V_{mn}(y) \cos\frac{m\pi}{l}x \cos\frac{n\pi}{b}z, \\
w(x, y, z) &= \sum_m \sum_n W_{mn}(y) \cos\frac{m\pi}{l}x \sin\frac{n\pi}{b}z, \\
p'(x, y, z) &= \sum_m \sum_n P_{mn}(y) \cos\frac{m\pi}{l}x \cos\frac{n\pi}{b}z.
\end{aligned} \tag{3.18}$$

Substituting (3.18) into eqns $(3.12)_2$, $(3.12)_3$, $(3.10)_1$ we obtain $m \times n$ ordinary differential equations

$$\begin{aligned}
b_2 V''_{mn} - (b_1\nu_m^2 + b_3\nu_n^2)V_{mn} - P'_{mn} &= 0, \\
c_2 W''_{mn} - (c_1\nu_m^2 + c_3\nu_n^2)W_{mn} + \nu_n P_{mn} &= 0, \\
V'_{mn} + \nu_n W_{mn} &= 0,
\end{aligned} \tag{3.19}$$

where

$$\nu_m \equiv \frac{m\pi}{l}, \qquad \nu_n \equiv \frac{n\pi}{b}. \tag{3.20}$$

Since for $n = 0$, the only solution is $w = 0$ we assume hereafter $n \neq 0$. From $(3.19)_2$, $(3.19)_3$ we find that

$$W_{mn} = -\frac{1}{\nu_n}V'_{mn}, \tag{3.21}$$

$$P_{mn} = \frac{1}{\nu_n^2}[c_2 V'''_{mn} - (c_1\nu_m^2 + c_3\nu_n^2)V'_{mn}]$$

and substituting (3.21) into (3.19) we obtain the fourth order equation for V_{mn}:

$$V_{mn}^{(\mathrm{IV})} - 2BV_{mn}'' + CV_{mn} = 0, \tag{3.22}$$

where

$$\begin{aligned} B &\equiv \nu_n^2\lambda^{-2}K(\lambda) + \tfrac{1}{2}\nu_m^2\lambda^{-2}(\Phi_1+\Phi_2)^{-1}[p - t - \lambda^{-2}(\Phi_2-\Phi_4)], \\ K(\lambda) &\equiv \tfrac{1}{2}(\lambda^2+\lambda^{-2}) + (\lambda^2-\lambda^{-2})^2(\Phi_1+\Phi_2)^{-1}(\Phi_{11}+2\Phi_{12}+\Phi_{22}), \\ C &\equiv \nu_n^4\lambda^{-4} + \nu_n^2\nu_m^2\lambda^{-2}(\Phi_1+\Phi_2)^{-1}[p - t - \lambda^2(\Phi_2-\Phi_4)]. \end{aligned} \tag{3.23}$$

The solution of (3.22), assuming $r_1 \neq r_2$, is

$$V_{mn}(y) = A_1e^{r_1y} + A_2e^{r_2y} + A_3e^{-r_1y} + A_4e^{-r_2y}, \tag{3.24}$$

where $\pm r_1$, $\pm r_2$ are the roots of the characteristic equation

$$r^4 - 2Br^2 + C = 0. \tag{3.25}$$

To determine the constants $A_1, \ldots, A_4$, we employ the boundary conditions $(3.15)_2$, which in view of (3.18), (3.21) take the form

$$\text{on} \quad y = \pm h: \qquad \begin{aligned} \nu_m V_{mn} &= 0, \\ V_{mn}'' + \nu_n^2 V_{mn} &= 0 \\ V_{mn}''' - (2B+\nu_n^2)V_{mn}' &= 0. \end{aligned} \tag{3.26}$$

The first condition is satisfied if

$$m = 0 \qquad \text{or} \qquad V_{mn}(\pm h) = 0. \tag{3.27}$$

These two cases lead to qualitatively different results.

4. BUCKLING IN THE PLANE OF ISOTROPY

Setting $m = 0$ in (3.18) implies $v_{,x} = w_{,x} = p'_{,x} = 0$ and since $u = 0$ it leads to the plane strain problem. The deformation takes place in the plane perpendicular to fibres, i.e. in the plane of material isotropy. In this case eqns (3.10), (3.12) and the boundary conditions (3.15) do not depend on t and Φ_4. It can be verified that they are, with exception of $(3.12)_1$, the same as in the analogous boundary value problem formulated for homogeneous, isotropic and incompressible material. The only difference concerns the first equation of equilibrium (3.12) which in our case is not identically satisfied and has the solution $T' = c(y, z)$. Consequently, the stress component τ'^{11}, like τ^{11} in the state B, is not determined by the plane strain solution and can be an arbitrary function of y, z.

Therefore, the solution of the problem for $m = 0$ (with the exception of the stress component in the direction of fibres, which is arbitrary and has no influence on the remaining solution) is identical with the corresponding plane strain solution for a material with a strain energy function of the form $W = W(I_1, I_2) - \frac{1}{2}p(I_3 - 1)$.

This solution has been derived by many authors for various forms of the strain energy function W. Let us mention the first paper by Wesołowski[4] and more recent by Sawyers and Rivlin[6]. The latter authors presented a general discussion of the bifurcation conditions for any elastic, incompressible material, provided $K(\lambda) \geq 1$, $K(\lambda)$ being defined in (3.23). Moreover in [6] one can find references to other papers dealing with the stability of a rectangular block under the assumption of plane strain.

Summarizing, the solution for $m = 0$, if it exists for a specified form of the strain energy function W, leads to the buckling in the plane of the material isotropy. Details concerning the buckling mode and the range of existence of the solution can be found in [4, 6] and other papers.

5. INTERNAL BUCKLING

Hereafter we assume $m \neq 0$ and from $(3.26)_1$ we have $V_{mn}(\pm h) = 0$ whence $v(\pm h) = 0$. According to (3.15) we have $u(\pm l) = 0$, $w(\pm b) = 0$; then the normal displacements vanish over the whole boundary and a solution for $m \neq 0$ represents a state of the body called by Biot[5] an internal buckling. Biot has shown that the internal buckling may occur in a homogeneous elastic medium of infinite extent or confined by rigid boundaries and has considered, as an example, a rectangular block confined between rigid boundaries. Postponing the problem of rigidly confined boundaries to the next section we show that in the case of inextensible, transversely isotropic material an internal buckling may occur in the presence of free surface.

Substituting (3.24) into the boundary conditions (3.26) we obtain a set of six homogeneous algebraic equations for four constants A_1, A_2, A_3, A_4

$$\begin{gathered}(A_1 - A_3) sh r_1 h = 0,\\ (A_2 - A_4) sh r_2 h = 0,\\ (A_1 - A_3) r_1 (r_2^2 + \nu_n^2) ch r_1 h + (A_2 - A_4) r_2 (r_1^2 + \nu_n^2) ch r_2 h = 0,\\ (A_1 + A_3) ch r_1 h = 0,\\ (A_2 + A_4) ch r_2 h = 0,\\ (A_1 + A_3) r_1 (r_2^2 + \nu_n^2) sh r_1 h + (A_2 + A_4) r_2 (r_1^2 + \nu_n^2) sh r_2 h = 0.\end{gathered} \tag{5.1}$$

A non-zero solution of (5.1) exists if and only if

$$r_1 = \frac{k_1 \pi}{h} i \quad \text{and} \quad \left(r_2 = \frac{k_2 \pi}{h} i \quad \text{or} \quad r_2 = \frac{n\pi}{b} i \right), \tag{5.2}$$

where k_1, k_2 are arbitrary integers, or they are integers plus $\frac{1}{2}$.

Substituting (5.2) and (3.23) into (3.25) we arrive at the following restrictions on parameters of the problem:

$$\begin{gathered}\kappa^2 = \frac{(\lambda^{-2} - \lambda^2)(\Phi_2 - \Phi_4)}{F(\lambda, \eta_1, \eta_2)(\Phi_1 + \Phi_2)},\\ t = -(\lambda^{-2} - \lambda^2)(\Phi_2 - \Phi_4) G(\lambda, \eta_1, \eta_2) + \lambda^2(\Phi_1 + \Phi_4) + \Phi_2,\end{gathered} \tag{5.3}$$

where

$$\begin{gathered}\kappa \equiv \frac{\nu_n}{\nu_m}, \qquad \eta_1 \equiv \frac{r_1}{i\nu_n}, \qquad \eta_2 \equiv \frac{r_2}{i\nu_n}\\ F(\lambda, \eta_1, \eta_2) \equiv 2K(\lambda) - \lambda^{-2} + \lambda^2(\eta_1^2 \eta_2^2 + \eta_1^2 + \eta_2^2),\\ G(\lambda, \eta_1, \eta_2) \equiv \frac{\lambda^2 \eta_1^2 \eta_2^2 - \lambda^{-2}}{F(\lambda, \eta_1, \eta_2)}\end{gathered} \tag{5.4}$$

We seek such values of the parameters κ, η_1, η_2 which satisfy (5.3) and (i) for a specified λ lead to the minimum value of $|t|$, (ii) for a specified t lead to the value of λ closest to unity. These two cases correspond to two processes of loading of the specimen. In the first case a specimen is uniformly deformed in the (y, z)-plane in the range of λ not causing the internal buckling for $t = 0$ and subsequently is loaded by pressure t. The question is whether this pressure has an influence on stability and if so, what is the minimum value of $|t|$ causing the instability. In the second case an undeformed specimen is loaded by pressure t in the range of t not causing the internal buckling for $\lambda = 1$ and next the uniform deformation in (y, z)-plane is introduced. This problem concerns the determination of the first λ which leads to the internal buckling in a specimen initially prestressed in the direction of fibres.

Since κ is a real number, $(5.3)_1$ has a solution if its right-hand side is non-negative. The sign of this expression depends on the properties of a material through the function $\Phi_K(\lambda)$ and can not be in general predicted.

Although we have $\Phi_1(\lambda)+\Phi_2(\lambda)>0$ as a consequence of the requirement that in the uniform compression $\tau^{22}>\tau^{33}$ causes $\lambda_2>\lambda_3$ (eqn 3.5), but we are unable to estimate the term $\Phi_2(\lambda)-\Phi_4(\lambda)$. Let us first consider the case

$$(\lambda^{-2}-\lambda^2)(\Phi_2-\Phi_4)\geq 0. \tag{5.5}$$

Under this assumption the sign of $\kappa^2(\lambda, \eta_1, \eta_2)$ depends only on the sign of $F(\lambda, \eta_1, \eta_2)$ and we can choose η_1, η_2 such that $F>0$ and then $\kappa^2(\lambda, \eta_1, \eta_2)\geq 0$.

The function $G(\lambda, \eta_1, \eta_2)$ except the case $F=0$ is a continuous function of η_1, η_2 and for $F\neq 0$ and

$$K(\lambda)\geq -1 \tag{5.6}$$

it is increasing in η_1, η_2. Since $\lim\limits_{\substack{\eta_1\to\infty\\ \eta_2\to\infty}} G(\lambda, \eta_1, \eta_2)=1$, we have

$$G(\lambda, \eta_1, \eta_2)\leq 1. \tag{5.7}$$

Observe, that $K(1)=1$ and (5.6) is satisfied in some vicinity of $\lambda=1$ for any material. Moreover, Sawyers and Rivlin[8] have shown, that for any elastic isotropic homogeneous material (5.6) is the necessary condition of material stability. It appears likely that it is also true for the transversely isotropic, inextensible material. However, this has not so far been proven and (5.6) should be treated as a restriction on the class of the considered materials.

In view of (5.5), (5.7) we find from $(5.3)_2$ the lower bound of $t(\lambda, \eta_1, \eta_2)$, namely

$$t(\lambda, \eta_1, \eta_2)\geq t_1(\lambda), \tag{5.8}$$

where

$$t_1(\lambda)\equiv -(\lambda^{-2}-\lambda^2)(\Phi_2-\Phi_4)+\lambda^2(\Phi_1+\Phi_4)+\Phi_2. \tag{5.9}$$

Since $t_1(\lambda)=\lim\limits_{\substack{\eta_1\to\infty\\ \eta_2\to\infty}} t(\lambda, \eta_1, \eta_2)$ and $\lim\limits_{\substack{\eta_1\to\infty\\ \eta_2\to\infty}} \kappa(\lambda, \eta_1, \eta_2)=0$ then $t(\lambda, \eta_1, \eta_2)\to t_1(\lambda)$ as $n/m\to 0$, $k_1/n\to\infty$, $k_2/n\to\infty$.

The function $t_1(\lambda)$ depends on λ through the material functions $\Phi_K(\lambda)$ and at this stage of generality not much can be said about the properties of this function. A hypothetical curve $t_1=t_1(\lambda)$ is drawn in Fig. 1, where the parameter λ is replaced by the parameter s such that $s=\lambda$ for $\lambda\geq 1$ or $s=\lambda^{-1}$ for $\lambda\leq 1$.

From elementary considerations for simple shearing and the requirement that a sign of the shear force must be compatible with the sign of the shear angle, it follows that

$$\phi_1+\phi_2+\phi_4>0. \tag{5.10}$$

Therefore in some vicinity of $\lambda=1$, $t_1(\lambda)>0$ and for these λ, $t=t_1(\lambda)$ is the required minimum value of $|t|$.

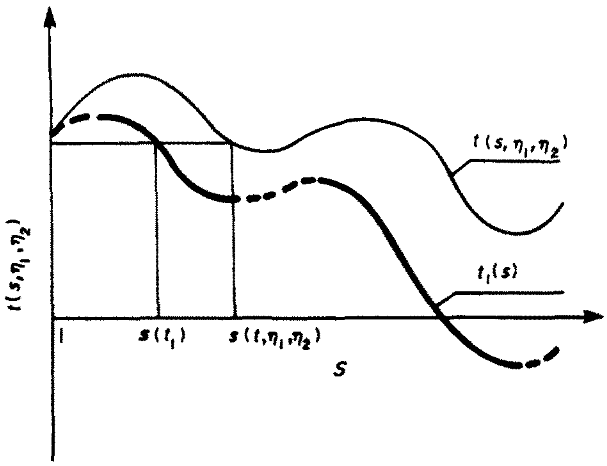

Fig. 1. Illustration of (5.8), (5.9). The pressure t vs parameter s; $s=\lambda$ for $\lambda\geq 1$ or $s=\lambda^{-1}$ for $\lambda\leq 1$.

If, for a specified material, the equation $t_1(\lambda)=0$ has real roots satisfying (5.5) then the roots closets to unity limit the range of initial deformation for $t=0$.

We arrive at the following conclusion: a specimen initially deformed without loosing its stability and subsequently loaded in the direction of fibres by a compressive pressure t, remains stable for $t<t_1(\lambda)$ while for $t\geq t_1(\lambda)$ an internal buckling takes place. We may say that the compressive pressure has a destabilizing influence on the stability of initially deformed specimen, while the tensile pressure has no influence at all.

In the case of a specimen initially prestressed in the direction of the fibres, the pressure t, which can be applied to the undeformed specimen without causing an internal buckling, is limited by the value of t_1 for $\lambda=1$, i.e. $t<t_1(1)$.

If, for a specified material $t_1(\lambda)\geq t_1(1)$ for each λ satisfying (5.5), then in view of (5.8), $t(\lambda,\eta_1,\eta_2)\geq t_1(1)$ and for $t<t_1(1)$ an internal buckling does not take place for any λ. A uniform deformation imposed on a prestressed specimen can cause the internal buckling if in some interval of λ, $t_1(\lambda)<t_1(1)$. Then there exists an interval of λ, such that $(\mathrm{d}t_1/\mathrm{d}s)<0$, where $s=\lambda$ for $\lambda\geq 1$ or $s=\lambda^{-1}$ for $\lambda\leq 1$ and in view of (5.8), $s(t_1)\leq s(t,\eta_1,\eta_2)$; see Fig. 1. The smallest s for which an internal buckling is possible, lies on a curve constructed of intervals of decreasing $t_1(s)$ shown by the full line in Fig. 1.

We may say that the initial compressive pressure $t>0$ has a destabilizing influence on the stability, since an increase of $|t|=t$ leads to a decrease of s, while the initial tensile pressure is stabilizing since an increase of $|t|=-t$ leads to an increase of s.

In the case

$$(\lambda^{-2}-\lambda^2)(\Phi_2-\Phi_4)<0 \tag{5.11}$$

the necessary condition of existence of a solution is $F<0$ and then

$$2K(\lambda)-\lambda^{-2}<0. \tag{5.12}$$

For any elastic material (5.12) is not satisfied in some vicinity of $\lambda=1$, since $K(1)=1$. Moreover, there are some materials for which (5.12) is not satisfied for any λ. For example, all materials such that $\Phi_{11}+2\Phi_{12}+\Phi_{22}\geq 0$ and among them the material with a strain energy function linear in the first and second invariant. However, if for a specified material there exists λ satisfying (5.11) we can find, for this λ, the minimum value of the pressure t which may cause the internal buckling.

For $F<0$ and $K(\lambda)\geq -1$, $G(\lambda,\eta_1,\eta_2)$ is also an increasing function of η_1, η_2 and $G(\lambda,0,0)=-[2\lambda^2K(\lambda)-1]^{-1}$; then

$$G(\lambda,\eta_1,\eta_2)>-\frac{1}{2\lambda^2K(\lambda)-1} \tag{5.13}$$

for each λ satisfying (5.11). In view of (5.3), (5.11), (5.13) we find that

$$t(\lambda, \eta_1, \eta_2) > t_2(\lambda), \tag{5.14}$$

where

$$t_2(\lambda) \equiv \frac{\lambda^{-2} - \lambda^2}{2\lambda^2 K(\lambda) - 1}(\Phi_2 - \Phi_4) + \lambda^2(\Phi_1 + \Phi_4) + \Phi_2. \tag{5.15}$$

A further discussion can be carried out similarly to the previous case.

Knowing the material functions $\Phi_K(\lambda)$ we can establish in the (t, λ)-plane the region in which the internal buckling becomes possible and this region is independent of the specimen dimensions. Besides, for specified dimensions, independently of t, there exist intervals of λ in which the buckling in the plane of isotropy is admissible, see Section 4. Therefore, the region of the unstable states consists of the sum of these two zones.

6. RIGIDLY CONFINED BOUNDARIES

Let us consider a rectangular block rigidly confined by smooth plates. The only difference, in comparison with the previously discussed boundary value problem, concerns the boundary conditions on the surface $y = \pm h$. Namely, instead of prescribing on this surface a normal stress, we prescribe a normal displacement, i.e. we disregard (3.14) and we replace in (3.15) the last condition on $y = \pm h$, $p' - (b_2 + c_2)v_{,y} = 0$ by

$$v(\pm h) = 0. \tag{6.1}$$

As a consequence the boundary conditions for the eqn (3.22) are

$$\begin{aligned} V_{mn}(\pm h) &= 0, \\ V''_{mn}(\pm h) &= 0. \end{aligned} \tag{6.2}$$

Substituting (3.24) into (6.2) we find that a solution exists iff

$$r_1 = \frac{k_1 \pi}{h} i \qquad \text{or} \qquad r_2 = \frac{k_2 \pi}{h} i. \tag{6.3}$$

Using the notation

$$\nu_k \equiv \frac{k\pi}{h} \tag{6.4}$$

we arrive at the following restriction on the parameters

$$t = p - (\Phi_2 - \Phi_4)\frac{\lambda^{-2}\nu_k^2 + \lambda^2 \nu_n^2}{\nu_k^2 + \nu_n^2} + (\Phi_1 + \Phi_2)\frac{\lambda^2 \nu_k^4 + 2K(\lambda)\nu_k^2 \nu_n^2 + \lambda^{-2}\nu_n^4}{\nu_m^2(\nu_k^2 + \nu_n^2)}. \tag{6.5}$$

Since, the last term in the r.h.s. is for $K(\lambda) \geq -1$ non-negative we find that for $(\lambda^{-2} - \lambda^2)(\Phi_2 - \Phi_4) \geq 0$

$$t(\lambda, p; \nu_k, \nu_m, \nu_n) \geq t_3(\lambda, p), \tag{6.6}$$

where

$$t_3(\lambda, p) \equiv p - \lambda^{-2}(\Phi_2 - \Phi_4) \tag{6.7}$$

and for $(\lambda^{-2}-\lambda^2)(\Phi_2-\Phi_4)\leq 0$

$$t(\lambda, p; \nu_k, \nu_m, \nu_n) \geq t_4(\lambda, p), \tag{6.8}$$

where

$$t_4(\lambda, p) \equiv p - \lambda^2(\Phi_2-\Phi_4). \tag{6.9}$$

$t \to t_3$ as $\dfrac{k}{n}\to\infty$, $\dfrac{k}{m}\to 0$ and $t\to t_4$ as $\dfrac{n}{k}\to\infty$, $\dfrac{n}{m}\to 0$.

For $\lambda = 1$, denoting $\tau^{33}=\tau^{22}=-\bar{p}$, we obtain

$$t_3 = t_4 = \bar{p} + \Phi_1(1) + \Phi_2(1) + \Phi_4(1).$$

It means that a compressive pressure $\bar{p} > 0$ has a stabilizing influence on the stability while a tensile pressure $\bar{p} < 0$ is destabilizing.

The considered model of ideal fibre-reinforced materials allows to apply an arbitrary pressure in the directions of fibres due to the assumption of inextensibility. However, a real material is usually not infinitely inextensible and moreover a matehmatical description of the fibres geometry is only an approximation to the real geometry. The procedure of superposing an arbitrary field of small displacements on finite strains can be treated as a matehmatical modelling of this imperfection of a real material. Then, the restrictions on loads which follow from this procedure should not be, in our opinion, neglected in engineering designs, especially in the case of materials with kinematical constraints.

REFERENCES

1. A. J. M. Spencer, *Deformations of fibre-reinforced materials*. Clarendon Press, Oxford (1972).
2. A. E. Green, R. S. Rivlin and R. T. Shield, General theory of small elastic deformations superposed on large elastic deformations. *Proc. Roy. Soc. A*, **211**, 128–154 (1952).
3. A. E. Green and A. J. M. Spencer, The stability of a circular cylinder under finite extension and torsion. *J. Math. Phys.* **37**, 316–338 (1959).
4. Z. Wesołowski, Stability in some cases of tension in the light of the theory of finite strain. *Arch. Mech. Stos.* **14**, 875–900 (1962).
5. M. A. Biot, Internal buckling under initial stress in finite elasticity. *Proc. Roy. Soc. A*, **273**, 306–328 (1963).
6. K. N. Sawyers and R. S. Rivlin, Bifurcation conditions for thick elastic plate under thrust. *Int. J. Solid Structures* **10**, 483–501 (1968).
7. Z. Wesołowski, *Dynamic problems of non-linear theory of elasticity*, PWN, Warsaw (1974). (In Polish).
8. K. N. Sawyers and R. S. Rivlin, Instability of an elastic material. *Int. J. Solids Structures* **9**, 607–613 (1973).